Problem 4.56

Coincident spectral lines.[72] According to the Rydberg formula (Equation 4.93) the wavelength of a line in the hydrogen spectrum is determined by the principal quantum numbers of the initial and final states. Find two distinct pairs $\{n_i, n_f\}$ that yield the *same* λ. For example, $\{6851, 6409\}$ and $\{15283, 11687\}$ will do it, but you're not allowed to use those!

Solution

The Rydberg formula is given in Equation 4.93 on page 155.

$$\frac{1}{\lambda} = \mathcal{R}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{4.93}$$

If there are two distinct pairs, $\{a, b\}$ and $\{c, d\}$, with the same λ, then they satisfy

$$\begin{cases} \frac{1}{\lambda} = \mathcal{R}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) \\ \frac{1}{\lambda} = \mathcal{R}\left(\frac{1}{d^2} - \frac{1}{c^2}\right) \end{cases}$$

$$\begin{cases} \frac{1}{\mathcal{R}\lambda} = \frac{1}{b^2} - \frac{1}{a^2} \\ \frac{1}{\mathcal{R}\lambda} = \frac{1}{d^2} - \frac{1}{c^2} \end{cases}.$$

Combine the equations into one and write it in the most convenient way.

$$\frac{1}{b^2} - \frac{1}{a^2} = \frac{1}{d^2} - \frac{1}{c^2}$$

$$\frac{a^2 - b^2}{a^2 b^2} = \frac{c^2 - d^2}{c^2 d^2}$$

$$c^2 d^2 (a^2 - b^2) = a^2 b^2 (c^2 - d^2)$$

This is the condition to test for that determines whether the two pairs, $\{a, b\}$ and $\{c, d\}$, yield the same λ.

The Easy Way

Switch the first entry of the first pair with the second entry of the second pair.

$$\{11687, 6409\} \text{ and } \{15283, 6851\}$$

The Hard Way

Write a computer program in your favorite language using loops to run through many integers and display the ones that satisfy the condition. It turns out that there are more and more distinct pairs the higher the numbers are; 85 is the highest we'll go here.

[72] Nicholas Wheeler, "Coincident Spectral Lines" (unpublished Reed College report, 2001).

The program here is written in C.

```
#include <stdio.h>

int main()
{
    int limit = 85;
    int a,b,c,d;

    for(a = 2; a <= limit; a++)
    {
        for(b = 1; b < a; b++)
        {
            for(c = 2; c < a; c++)
            {
                for(d = 1; d < c; d++)
                {
                    if(c*c*d*d*(a*a - b*b) == a*a*b*b*(c*c - d*d) && b != d && a != c)
                        printf("{%d,%d} and {%d,%d}\n", a,b,c,d);
                }
            }
        }
    }

    return 0;
}
```

Its output is shown below.

```
{35,7} and {7,5}
{55,11} and {22,10}
{55,22} and {11,10}
{56,8} and {14,7}
{56,14} and {8,7}
{63,18} and {21,14}
{63,21} and {18,14}
{70,14} and {14,10}
{72,8} and {9,6}
{72,9} and {8,6}
{80,45} and {48,36}
{80,48} and {45,36}
{85,34} and {51,30}
{85,51} and {34,30}
```